ADDENDUM TO ENDING LAMINATIONS AND CANNON-THURSTON MAPS: PARABOLICS

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Abstract. In earlier work, we had shown that Cannon-Thurston maps exist for Kleinian punctured surface groups without accidental parabolics. In this note we prove that pre-images of points are precisely end-points of leaves of the ending lamination whenever the Cannon-Thurston map is not one-to-one. This extends earlier work done for closed surface groups.

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1. Ingredients

In [Mj06] we proved the existence of Cannon-Thurston maps for surface groups without accidental parabolics. For closed surface groups, we described the structure of these maps in terms of ending lainations in [Mj07]. In this note we extend the results of [Mj07] to punctured surfaces.

1) Equivalence Relations on $\mathbb{S}^1$:
Suppose that a group G acts on S^1 preserving a closed equivalenve relation $\mathcal{R}$. An example $\mathcal{R}_{\mathcal{L}}$ of such a relation comes from a lamination $\mathcal{L}$, where two points on S^1 are equivalent if they are end-points of a leaf of $\mathcal{L}$. The equivalence relation $\mathcal{R}_{\mathcal{L}}$ is obtained as the transitive closure of this relation.

Definition 1.1. [Bow07] *Two disjoint subsets, $P, Q \subset S^1$ are linked if there exist linked pairs, $\{x, y\} \subset P$ and $\{z, w\} \subset Q$. $\mathcal{R}$ is unlinked if distinct equivalence classes are unlinked.*

The following Lemma due to Bowditch give us a way of recognising relation coming from laminations.

Lemma 1.2. *(Lemma 9.2 of* [Bow07]*) Let $\mathcal{R}$ be a non-empty closed unlinked G-invariant equivalence relation on S^1. Suppose that no pair of fixed points of any loxodromics are identified under $\mathcal{R}$. Then there is a unique complete perfect lamination, $\mathcal{L}$, on S such that $\mathcal{R} = \mathcal{R}_{\mathcal{L}}$.*

2) Partial Electrocution
Let Y be the convex core of a simply (resp. doubly) degenerate 3-manifold of the form $S \times J$, where $J = [0, \infty)$ (resp. $\mathbb{R}$). Let $\mathcal{B}$ denote the equivariant collection of horoballs in $\widetilde{Y}$ covering the cusps of Y. Let X denote Y minus the interior of the horoballs in $\mathcal{B}$. Let $\mathcal{H}$ denote the collection of boundary horospheres.Then each $H \in \mathcal{H}$ with the induced metric is isometric to a Euclidean product $\mathbb{R} \times J$. Partially electrocute each $H \in \mathcal{H}$ by giving it the product of the zero metric (in the $\mathbb{R}$-direction) with the Euclidean metric (in the J-direction), The resulting space is

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quasi-isometric to what one would get by gluing to each H the mapping cylinder of the projection of H onto the J-factor. Let $\mathcal{J}$ denote the collection of copies of J obtained in this latter construction and let $(\mathcal{PEY}, d_{pel})$ denote the resulting partially electrocuted space. (See [MP07] for a more general discussion.) We have the following basic Lemma.

Lemma 1.3. [MP07] *$(\mathcal{PEY}, d_{pel})$ is a hyperbolic metric space and the sets $J_\alpha \in \mathcal{J}$ are uniformly quasiconvex.*

3) Split Geometry and Ladders

As pointed out in [Mj06] there exist a sequence of surfaces exiting the end(s) of Y, such that after removing the cusps and partially electrocuting them, and subsequently electrocuting split blocks, we obtain a model of split geometry. This was utilized in [Mj06] to obtain the structure of Cannon-Thurston maps for closed surface groups without accidental parabolics. The construction of split geometry recalled in the main body of [Mj06] goes through for punctured surfaces with the pro viso that we first partially electrocute the space. We refer to [Mj06] for details. Let S_i be the sequence of truncated surfaces (i.e. surfaces minus cusps) exiting the end and forming boundaries of the blocks in the model of split geometry. Note that after partial electrocution, the induced metric on the boundary horocycles of each $\widetilde{S_i}$ is the zero metric. Equivalently each horocycle HC is coned off to the corrsponding point of a $J_\alpha \in \mathcal{J}$. Let $(\widetilde{S_i}, d_{el})$ denote the resulting electric metrics.

From a geodesic $\lambda = \lambda_0 \subset (\tilde{S}_0, d_{el}) \subset \mathcal{PEY}$ we constructed in [Mj06] a 'hyperbolic ladder' $\mathcal{L}_\lambda$ such that $\lambda_i = \mathcal{L}_\lambda \cap \tilde{S}_i$ is an electro-ambient quasigeodesic in the (path) electric metric on $\tilde{S}_i$ induced by the graph metric d_G on $\mathcal{PEY}$.

We also constructed a large-scale retract $\Pi_\lambda : \mathcal{PEY} \to \mathcal{L}_\lambda$ such that the restriction π_i of Π_λ to $\tilde{S} \times \{i\}$ is, roughly speaking, a nearest-point retract of $\tilde{S} \times \{i\}$ onto λ_i in the (path) electric metric on $\tilde{S}_i$.

We have the following basic theorem from [Mj06].

Theorem 1.4. [Mj06] *There exists $C > 0$ such that for any geodesic $\lambda = \lambda_0 \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, the retraction $\Pi_\lambda : \mathcal{PEY} \to \mathcal{L}_\lambda$ satisfies:*

Then $d_G(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_G(x, y) + C$.

4) qi Rays

We also have the following from [Mj06].

Lemma 1.5. *There exists $C \geq 0$ such that for $x_i \in \lambda_i$ there exists $x_{i-1} \in \lambda_{i-1}$ with $d_G(x_i, x_{i-1}) \leq C$. Similarly there exists $x_{i+1} \in \lambda_{i+1}$ with $d_G(x_i, x_{i+1}) \leq C$. Hence, for all n and $x \in \lambda_n$, there exists a C-quasigeodesic ray r such that $r(i) \in \lambda_i \subset \mathcal{L}_\lambda$ for all i and $r(n) = x$.*

Hence, given $p \in \lambda_i$ the sequence of points $x_n, n \in \mathbb{N} \cup \{0\}$ (for simply degenerate groups) or $n \in \mathbb{Z}$ (for totally degenerate groups) with $x_i = p$ gives by Lemma 1.5 above, a quasigeodesic in the d_G-metric. Such quasigeodesics shall be referred to as *d_G-quasigeodesic rays.* Recall the following Proposition from [Mj07] in this context.

Proposition 1.6. *Let μ, λ be two bi-infinite geodesics on $\widetilde{S}$ such that $\mu \cap \lambda \neq \emptyset$. Then $\mathcal{L}_\lambda \cap \mathcal{L}_\mu = r$ is a quasigeodesic ray in $(\mathcal{PEY}, d_G)$..*

5) Easy Direction: Ideal points are identified by Cannon-Thurston Maps
The easy direction of the main Theorem 2.3 of this appendix is the same as that in [Mj07].

Proposition 1.7. *Let u, v be either ideal end-points of a leaf of a lamination, or ideal boundary points of a complementary ideal polygon. Then $\partial i(u) = \partial i(v)$.*

As in [Mj07] a **CT leaf** λ_{CT} will be a bi-infinite geodesic whose end-points are identified by ∂i.
An **EL leaf** λ_{EL} is a bi-infinite geodesic whose end-points are ideal boundary points of either a leaf of the ending lamination, or a complementary ideal polygon. Λ will denote the ending lamination for a simply degenerate group.

It remains to show that
• **A** *CT leaf* **is an** *EL leaf.*

Let $\mathcal{R}_{CT}$ denote the equivalence relation on S^1 induced by the Cannon-Thurston map for the punctured surface (existence proven in [Mj06]. By Proposition 1.7 pairs of end-points of leaves of Λ are contained in $\mathcal{R}_{CT}$. Hence, for simply degenerate groups, it suffices to show that $\mathcal{R}_{CT}$ is induced by a lamination (since no other lamination can properly contain Λ). By Lemma 1.2 it suffices to show that $\mathcal{R}_{CT}$ is unlinked. This is the content of the next section.

2. $\mathcal{R}_{CT}$ is unlinked

We adapt Corollary 2.7 of [Mj07] to the present context.

Proposition 2.1. [Mj07] **CT leaves have infinite diameter**
Let $\lambda_+(\subset \lambda \subset \tilde{S} \times \{0\} = \tilde{S}) = \mathbb{H}^2$ be a semi-infinite geodesic (in the hyperbolic metric *on $\tilde{S}$) contained in a CT leaf λ. Further suppose that λ_+ does not have a parabolic as its limit point in $\partial\mathbb{H}^2$. Then $dia_G(\lambda_+)$ is infinite, where dia_G denotes diameter in the graph metric restricted to $\widetilde{S}$.*

Proposition 2.2. *Let $i : \widetilde{S} \to \widetilde{M}$ be an inclusion of the universal cover of a punctured surface into the universal cover of the convex core M of a simply degenerate 3-manifold. Let ∂i be the associated Cannon-Thurston map. If λ is a CT-leaf in $\widetilde{S}$, $\mathcal{L}_\lambda$ the corrsponding ladder, and $r = r(n) \subset \mathcal{L}_\lambda$ a qi ray, then there exists $z \in \partial\widetilde{M}$ such that $r(n) \to z = \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$ as $n \to \infty$.*

Proof: We first observe that both end-points $\lambda_{-\infty}, \lambda_\infty$ of the CT leaf λ cannot be parabolics. For then they would have to be base points of *different* horoballs in $\widetilde{M}$ as they correspond to different lifts of the cusp(s) of M.
Case 1: Both $\lambda_{-\infty}, \lambda_\infty$ are non-parabolic.
The proof of Proposition 2.13 of [Mj07] goes through in this context mutatis mutandis.
Case 2: Exactly one of $\lambda_{-\infty}, \lambda_\infty$ is a parabolic.
Without loss of generality assume that $\lambda_{-\infty}$ is a parabolic. Let B be the horoball in $\widetilde{M}$ based at $w = \partial i(\lambda_{-\infty})$ and let H be the horosphere boundary of B. Let o be the point of intersection of λ with H. Choose a sequence of points $a_n, b_n \in \lambda$ such that $a_n \to \lambda_-\infty$ and $b_n \to \lambda_\infty$. Let $\overline{a_n b_n}$ be the geodesic in $\widetilde{M}$ joining a_n, b_n. Then by the existence of Cannon-Thurston maps for $i : \widetilde{S} \to \widetilde{M}$ it follows easily (see Lemma 2.1 of [Mit98] for instance) that there exists a function $M(n) \to \infty$ as

$n \to \infty$ such that $\overline{a_n b_n}$ lies outside $B_{M(n)}(o) \subset \widetilde{M}$. Hence, if $q_n = \overline{a_n b_n} \cap H$ then $d(q_n, o) \geq M(n)$ and the geodesic subsegment $\overline{q_n b_n}$ lies outside $B_{M(n)}(o) \subset \widetilde{M}$.

Let $N = \widetilde{M} \setminus \bigcup_\alpha B_\alpha$ be the complement of open horoballs and d_G be the graph metric on N obtained after first partially electrocuting horospheres. Let (q_n, b_n) be the geodesic joining q_n, b_n in (N, d_G). Then by weak relative hyperbolicity of N, (see Lemma 2.1 of [Mj06]) (q_n, b_n) and $\overline{q_n b_n}$ lie in a bounded neighborhood of each other in (N, d_G).

By Proposition 2.1 $d_G(o, b_n) \to \infty$ as $n \to \infty$. Also, $d_G(o, q_n)$ is the number of vertical blocks between o and q_n and hence $d_G(o, q_n) \to \infty$. But $a_n \to \lambda_-\infty$ implies $q_n \to \infty$. Hence $d_G(o, q_n) \to \infty$ as $a_n \to \lambda_-\infty$. Finally (see for instance the argument in Sections 6.3, 6.4 of [Mj06]) there exists a function $M_1(n) \to \infty$ as $n \to \infty$ such that (q_n, b_n) lies outside $B_{M_1(n)}(o) \subset (N, d_G)$.

Now recall that $\Pi_\lambda : N \to \mathcal{L}_\lambda$ is a coarse Lipschitz retract by Theorem 1.4. Hence $\Pi_\lambda[(q_n, b_n)] \subset \mathcal{L}_\lambda$ uniform quasigeodesic in (N, d_G).

Further, since $q_n \in H$ and since Π_λ essentially fixes the horosphere H, it follows that $d_G(\Pi_\lambda(q_n), q_n) \leq 1$. Also $\Pi_\lambda(b_n) = b_n$. Therefore there exists a function $M_2(n) \to \infty$ as $n \to \infty$ such that $\Pi_\lambda[(q_n, b_n)]$ lies outside $B_{M_2(n)}(o) \subset (N, d_G)$.

Next, since $H \cap \mathcal{L}_\lambda$ and b_n lie on opposite sides of the qi ray $r = r(n) \subset \mathcal{L}_\lambda$ it follows that there exists $z_n \in (q_n, b_n)$ such that $d_G(z_n, r)$ is uniformly bounded.

Also there exists $t_n \in \overline{q_n b_n}$ such that $d_G(z_n, t_n)$ and hence $d_G(t_n, r)$ is uniformly bounded.

Since $t_n \in \overline{q_n b_n}$ it follows that $t_n \to \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$. Since $d_G(t_n, r)$ is uniformly bounded, there exists $s_n \in r$ such that $d_G(t_n, s_n)$ is uniformly bounded and therefore t_n, s_n are separated by a uniformly bounded number of split components. By uniform graph quasiconvexity of split components (Definition-Theorem 1.6 of [Mj07]) it follows that $s_n \to \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$. Finally if r_{s_n} denotes the part of the ray r 'above' s_n, (i.e. $[s_n, \infty)$) then joining points of r_{s_n} in successive blocks by hyperbolic geodescics we obtain a path σ_n which contains a semi-infinite hyperbolic ray (the limit of hyperbolic geodesic segments joining s_n to points arbitrarily far along r) (by weak relative hyperbolicity of N, – Lemma 2.1 of [Mj06]). Hence $r(n) \to \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$ as $n \to \infty$. $\square$

Theorem 2.3. *Let $\partial i(a) = \partial i(b)$ for $a, b \in S^1_\infty$ be two distinct points that are identified by the Cannon-Thurston map corresponding to a simply degenerate surface group (without accidental parabolics). Then a, b are either ideal end-points of a leaf of the ending lamination (in the sense of Thurston), or ideal boundary points of a complementary ideal polygon. Conversely, if a, b are either ideal end-points of a leaf of a lamination, or ideal boundary points of a complementary ideal polygon, then $\partial i(a) = \partial i(b)$.*

Suppose λ and μ are intersecting CT leaves, i.e. $\partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$ and $\partial i(\mu_{-\infty}) = \partial i(\mu_\infty)$.

Consider the ladders $\mathcal{L}_\lambda$ and $\mathcal{L}_\mu$. Let $r(i) = \lambda_i \cap \mu_i$ be a quasigeodesic ray as per Proposition 1.6. By Proposition 2.2, r converges to a point z on $\partial\widetilde{M}$ such that $z = \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty) = \partial i(\mu_{-\infty}) = \partial i(\mu_\infty)$. Hence if $\{a, b\}, \{c, d\} \in \mathcal{R}_{CT}$, then either $\{a, b, c, d\}$ are all mutually related in $\mathcal{R}_{CT}$, or $\{a, b\}, \{c, d\}$ are unlinked. By Lemma 1.2, $\mathcal{R}_{CT}$ is induced by a lamination Λ_{CT}. By Proposition 1.7, the ending lamination Λ_{EL} is contained in Λ_{CT}. Since Λ_{EL} is filling and arational, it follows that $\Lambda_{EL} = \Lambda_{CT}$. $\square$

The modifications necessary to pass from the simply degenerate case to the totally degenerate case are exactly as in the last subsection of [Mj07].

References

[Bow07] B. H. Bowditch. The Cannon-Thurston map for punctured surface groups. *Math. Z. 255*, pages 35–76, 2007.

[Mit98] Mahan Mitra. Cannon-Thurston Maps for Trees of Hyperbolic Metric Spaces. *Jour. Diff. Geom.48*, pages 135–164, 1998.

[Mj06] Mahan Mj. Cannon-Thurston Maps for Surface Groups. *preprint, arXiv:math.GT/0607509*, 2006.

[Mj07] Mahan Mj. Ending Laminations and Cannon-Thurston Maps . *preprint, arXiv:math.GT/0702162*, 2007.

[MP07] Mahan Mj and Abhijit Pal. Relative Hyperbolicity, Trees of Spaces and Cannon-Thurston Maps. *arXiv:0708.3578, submitted to Geometriae Dedicata*, 2007.

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